Multiple beams from beacons

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Abstract

The math behind creating multiple beams from a phased array beacon. We compute the tradeoff between number of transmitters, number of beams, and beam quality. We find the expected amplitude, power and phase noise. We determine the degradation if the transmitters are all equal and fixed power.

1 Intro

Assume we have N transmitters, and want to generate M beams. Each transmitter is 1 watt (either absolute, or on the average, depending on the model). We assume the beams are randomly located on the sky. We assign the phases and amplitudes of each transitter as follows. For each target, compute the desired phase at the transmitter. Then for each transmitter, add (as vectors) the desired phases for each target, with amplitude $1/\sqrt{M}$. In the variable amplitude model this gives us both our amplitude and our phase. In the constant power model we then keep the phase and set the amplitude to 1. In the variable amplitude case the power will average 1, since the mean of the square of M random vectors of length $1/\sqrt{M}$ is $\sqrt{M}/\sqrt{M} = 1$.

2 Math

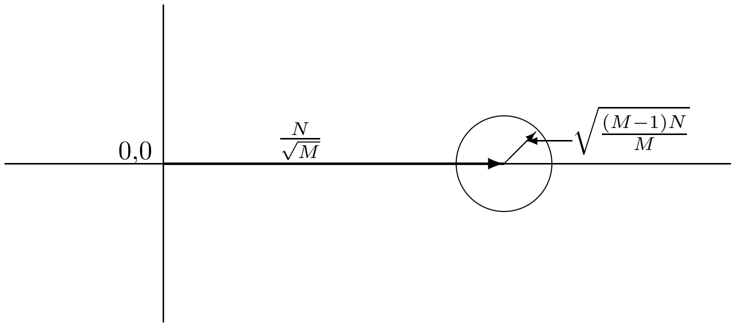
We use a result first derived by Rayleigh[1]. If we start at the origin, and make n steps of unit length in random directions, then the odds of finding the endpoint between r and $r + \mathrm{d}r$ from the origin is:

$$\frac{2}{n} r e^{-r^2/n} \mathrm{d}r$$

. The mean squared displacement is n, leading to the well known result that the average displacement grows like $\sqrt{n}$. The X or Y component, considered alone, has mean 0 and variance $n/2$.

3 Calculations

We look at the result as seen by one receiver. To do this, we add add transmitter voltages with the phase shift appropriate for the receiver direction. The $1/\sqrt{M}$ contribution computed from that reciever will add up in phase. The other $(M-1)N$ contributions will appear as a sum of random vectors, and will have magnitude roughly $\sqrt{\frac{(M-1)N}{M}}$. Thus the final voltage in the phase plane as seen by that receiver looks like this:



Although drawn as a circle, since it's statistical the error is really a cloud. All we know is that the voltage vector ends somewhere in that area. Therefore the main characteristics of a beam are as follows:

The expected amplitude is

$$\text{Amplitude}(M,N) = A_0 \frac{N}{\sqrt{M}}$$

where A_0 is the amplitude induced by a single transmitter. Equivalently the EIRP per beam is

$$\text{EIRP}(M,N) = P_0 \frac{N^2}{M}$$

where P_0 is the EIRP of a single transmitter. To find the expected variance in voltage, we want only the X component of the cloud. This will be $\frac{1}{M}\frac{(M-1)N}{2}$, and the standard deviation will be the square root of this. The Y component alone will have the same variance. So we have the relative uncertainty in amplitude

$$\text{RelAmpUncertainty}(M,N) = \sqrt{\frac{M-1}{2N}}$$

and the uncertainy in phase (in radians) will have the same numerical value. In degrees this is

$$\text{RelPhaseUncertainty}(M,N) = \sqrt{\frac{M-1}{2N}}\frac{180}{\pi}$$

4 Theory and Experiment

I wrote a simple program to test this model. This program allows the user to try N transmitters and M beams. This test case is 2 dimensional, which should make no difference. Each transmitter has an amplitude of 1 in the constant power case, and an average squared amplitude of 1 in the variable power case. We assume all the targets are far enough away so that the angles to the transmitters are the same. The transmitters are randomly located between $-N/2$ and $N/2$ meters at an average density of 1 per meter. The beam angles are randomly chosen between ± 45 degrees of the zenith. The wavelength is randomly chosen to be 0.0567 . The results are:

M	N	Amplitude		Ampl. Variation		Phase Variation	
		Theory	Exp	Theory	Exp	Theory	Exp
10	500	158.1	154.2	15	9.85	5.44	5.05
10	5000	1581	1531	47.4	73.9	1.71	1.93
10	50000	15811	15772	150	177	0.54	0.47
10	500000	158114	158231	474	438	0.17	0.14
100	5000	500	496	49.75	50.04	5.70	5.58
100	50000	5000	4979	157.3	159.6	1.80	1.82
100	500000	50000	49928	497.5	496.1	0.57	0.53
1000	50000	1581	1578	158	161	5.72	5.65
1000	500000	15811	15826	500	646	1.81	1.86

The agreement of theory and experiment for large M, values 100 or greater, is excellent. The agreement for small M is OK but not great. This is due to two causes - the real distribution is not as Rayleigh predicts for M as small as 10 (his solution is only for large M), and the distribution cannot be measured well with only 10 samples.

5 Constant power transmitters

It complicates the design considerably if each transmitter has to be able to control its amplitude as well as its phase. How much do we give up with constant amplitude? First, we need to find the expected amplitude at each transmitter. We start with the known vector and treat the remaining $M-1$ vectors as a random walk. We get a diagram that looks like figure 1, except that the displacement from the origin is $1/\sqrt{M}$ and the circular error region now has radius $\sqrt{\frac{M-1}{M}}$, and is now much larger than the distance to the origin. This can be thought of as the random walk of radius nearly 1 (actually $\sqrt{\frac{M-1}{M}}$ with the origin shifted slightly by $1/\sqrt{M}$). Note that this distribution, and the distribution after clipping, depends only on the value of M; the variation with

N is taken care of by summing N of these distributions.

When we keep the phase, but make the power constant, it's the same as mapping every point in the phase plane along a radial onto the single point where that radial intersects the unit circle. We can treat this as follows.

The random part of the distribution consists of M-1 steps of size $1/\sqrt{M}$. From Rayleigh, the odds of finding this in the ring between distance r and $r + \mathrm{d}r$ is

$$\frac{2M}{M-1} e^{-r^2 \frac{M}{M-1}} r \mathrm{d}r$$

To get the area density, we divide by the area of the ring to get

$$\text{Density}(r) = \frac{1}{\pi} \frac{M}{M-1} e^{-r^2 \frac{M}{M-1}}$$

We want the expected relative value of the X component of the resulting vector. This is computed by the following equation.

$$\frac{\int_0^{2\pi} \cos(\theta) \lim_{M \to \infty} \left[\frac{1}{\pi} \frac{M}{M-1} \int_0^\infty e^{-(r(q,\theta))^2 \frac{M}{M-1}} q \, \mathrm{d}q \right] \mathrm{d}\theta}{1/\sqrt{M}}$$

Where $r(q, \theta)$ is the distance from the center of the distribution

$$r(q, \theta) = \sqrt{(q \cos\theta - 1/\sqrt{M})^2 + (q \sin\theta)^2}$$

The part of the formula in square brackets is the probability that the phase is θ. This turns out to be a complicated expression which can be simplified if we only consider large M, which explains the limit. The outer integral simply averages over all angles, taking the contribution to the X component times the probability that angle is found. Finally, we take the ratio to $1/\sqrt{M}$ since that's the expected value in the variable power case.

We start by expanding the expression for r and using $\sin^2\theta + \cos^2\theta = 1$ to get

$$r^2(q, \theta) = q^2 - \frac{2}{\sqrt{M}} q \cos\theta + \frac{1}{M}$$

and so the inner probability becomes

$$\frac{1}{\pi} \frac{M}{M-1} \int_0^\infty e^{-(q^2 - \frac{2}{\sqrt{M}} q \cos\theta + \frac{1}{M}) \frac{M}{M-1}} q \, \mathrm{d}q$$

Since we are only interested in M large, we can drop the terms of $1/M$ which are the square of the terms with $1/\sqrt{M}$. To this order, $M/(M-1) = 1$, so we can drop these terms as well, to get

$$\frac{1}{\pi} \int_0^\infty e^{-(q^2 - \frac{2}{\sqrt{M}} q \cos\theta)} q \, \mathrm{d}q$$

which can be re-written as

$$\frac{1}{\pi}\int_0^\infty e^{-(q^2)}e^{\frac{2}{\sqrt{M}}q\cos\theta)}q\,\mathrm{d}q$$

We expand the second exponential into a power series and keep only the first term to get:

$$\frac{1}{\pi}\int_0^\infty e^{-(q^2)}(1+\frac{2}{\sqrt{M}}q\cos\theta)q\,\mathrm{d}q$$

The alert reader (if any) might well ask if this is legitimate, since the missing terms are multiplied by q, which ranges to infinity. However, here we are saved by the e^{-q^2} term, which tends to 0 even more rapidly, so for a sufficiently large M this will be OK. This gives us

$$\frac{1}{\pi}\int_0^\infty qe^{-q^2}\,\mathrm{d}q+\frac{1}{\pi}\frac{2\cos\theta}{\sqrt{M}}\int_0^\infty q^2e^{-q^2}\,\mathrm{d}q$$

The first term will not contribute to the outer integral, since it has no dependence on θ and hence integrates to 0 over one whole cycle. This leaves

$$\frac{1}{\pi}\frac{2\cos\theta}{\sqrt{M}}\int_0^\infty q^2e^{-q^2}\,\mathrm{d}q$$

This is a well known definite integral, value $\sqrt{(\pi)}/4$, leading to

$$\frac{1}{\sqrt{M}}\frac{\cos\theta}{2\sqrt{\pi}}$$

This is the non-uniform part of the probability that the phase is θ. Now we integrate over all phases. If the phase is θ, the contribution to the phase sum is $\cos\theta$, so we multiply the contribution times the probability of that contribution to get

$$\frac{1}{\sqrt{M}}\frac{1}{2\sqrt{\pi}}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$$

but $\cos^2\theta = 1/2 + 1/2\sin(2\theta)$ has an average value of $1/2$, so the last integral has a value of π, leading to an expected output voltage of

$$\frac{1}{\sqrt{M}}\frac{\sqrt{\pi}}{2}$$

We compare this to the expected value with no clipping, $1/\sqrt{M}$, to get the final relative amplitude of

$$\frac{\sqrt{\pi}}{2}$$

To check the previous calculation, we can also evaluate the clipping process numerically, and compared this to the analytical result with no clipping. We do this for various values of M:

M	multiplier
4	0.94603
8	0.915935
16	0.901281
32	0.894021
64	0.890361
128	0.888486
256	0.887506
65536	0.886277
1048576	0.886242

It certainly appears reasonable that this value is converging to $\sqrt{\pi}/2 = 0.88622692545275801364$ for large enough M, and the rate of convergence is right ($1/\sqrt{M}$ is about 1000 for the largest cases, and the result differs from the analytical limit by roughly 1 part in 1000). So we'll assume $\sqrt{\pi}/2$ as the expected value, and then the EIRP is about $\pi/4$, or about 0.7853 of the variable amplitude case. This is a loss of about 1.049 db.

The expected amplitude is then

$$\text{Amplitude}(M,N) = \frac{\sqrt{\pi}}{2} A_0 \frac{N}{\sqrt{M}}$$

where A_0 is the amplitude induced by a single transmitter. Equivalently the EIRP per beam is

$$\text{EIRP}(M,N) = \frac{\pi}{4} P_0 \frac{N^2}{M}$$

where P_0 is the EIRP of a single transmitter. To find the expected variance in voltage, we now have the sum of the X component of N unit vectors oriented at random. This variance will be $0.5N$, and the standard deviation will be the square root of this. The Y component alone will have the same variance. So we have the relative uncertainty in amplitude

$$\text{RelAmpUncertainty}(M,N) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{M}{2N}}$$

and the uncertainy in phase (in radians) will have the same numerical value. In degrees this is

$$\text{RelPhaseUncertainty}(M,N) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{M}{2N}} \frac{180}{\pi}$$

This simple model gives the following results.

M	N	Amplitude		Ampl. Variation		Phase Variation	
		Theory	Exp	Theory	Exp	Theory	Exp
10	500	140.1	137.4	15.8	12	6.46	8.16
10	5000	1401	1388	50	53	2.04	2.09
10	50000	14013	14086	158	208	0.65	0.47
10	500000	140125	141057	500	458	0.20	0.16
100	5000	443	441	50	51.6	6.47	6.15
100	50000	4431	4427	158	157	2.04	2.15
100	500000	44311	44310	500	533	0.65	0.73
1000	50000	1401	1401	158	163	6.47	6.49
1000	500000	14013	14016	500	611	2.04	2.03

Once again the agreement is excellent for large M and OK for small M.

6 Phase quantization

What if the phase is quantized? If there are P possible phases, then the nearest available phase will be uniformly distributed across $\pm 2\pi/P$. Thus the average contribution in the desired direction is

$$\frac{\int_{-\pi/P}^{\pi/P} \cos\theta \, d\theta}{2\pi/P}$$

This reduces to

$$\frac{P}{\pi}\sin(\frac{\pi}{P})$$

The following table shows the degradation, as a multiplier from the best case of arbitrary phase availability.

Number of phases, P	amplitude	power	db
2	0.636620	0.405285	-3.922398
3	0.826993	0.683918	-1.649960
4	0.900316	0.810569	-0.912098
6	0.954930	0.911891	-0.400572
8	0.974495	0.949641	-0.224405
12	0.988616	0.977361	-0.099448
16	0.993587	0.987215	-0.055883
32	0.998394	0.996791	-0.013957
64	0.999598	0.999197	-0.003488

7 Evenly spaced transmitters

What if the transmitters are evenly spaced? Then the signals will presumably be more correlated, since for each direction you are really picking just two random variables (initial phase and delta) instead of N. The following experiment shows this effect for variable power:

M	N	Amplitude		Ampl. Variation		Phase Variation	
		Theory	Exp	Theory	Exp	Theory	Exp
10	500	158.1	175.9	15	38	5.44	10.06
10	5000	1581	1543	47	82	1.71	0.56
10	50000	15811	15770	150	87	0.54	0.13
10	500000	158114	158088	474	52	0.17	0.01
100	5000	500	511	49	65	5.70	5.91
100	50000	5000	5072	157	506	1.80	0.44
100	500000	50000	50101	497	717	0.57	0.04
1000	50000	1581	1578	158	149	5.72	4.32
1000	500000	15811	15789	500	490	1.81	0.50

Here's the same experiment using constant power:

M	N	Amplitude		Ampl. Variation		Phase Variation	
		Theory	Exp	Theory	Exp	Theory	Exp
10	500	140.1	149.9	15.8	33.8	6.46	10.2
10	5000	1401	1394	50	83	2.04	1.52
10	50000	14013	14100	158	171	0.65	0.28
10	500000	140125	140978	500	321	0.20	0.06
100	5000	443	448	50	66	6.47	6.47
100	50000	4431	4469	158	463	2.04	0.97
100	500000	44311	44377	500	673	0.65	0.30
1000	50000	1401	1400	158	149	6.47	5.26
1000	500000	14013	14001	500	489	2.04	1.10

Except in the smallest cases, the variance in the amplitude seems to be a little more than predicted by the random model, and the variance in phase somewhat less. The random model still provides a fairly good prediction, however.

8 Extensions

Consider polarization.

References

[1] Rayleigh's 1880 article